

A HAMILTONIAN MODEL FOR HARD INELASTIC HADRONIC COLLISIONS

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Abstract

A Hamiltonian eikonal model for multiple production in high energy hadron-hadron collisions is presented and worked out with the aim of providing a simple frame for various different observables.

An important role is played by unitarity which is built in by construction in the Hamiltonian formulation. The eikonal approximation allows both a very effective simplification of the dynamics, and facilitates the discussion on the relevance of possible spatial inhomogeneities of the hadrons. The model is intended to describe only the hard interaction of the constituents, the structure of the incoming hadrons and the final hadronization processes are outside the scope of the present investigation.

1. Introduction

A Hamiltonian model for the description of the multiple production process in hadron-hadron collision is presented with the main aim of bringing together different observables within a unique frame. A particular attention is given to those features of the inelastic processes that can give informations on the proton structure. among these the relation between the hard-inelastic cross section and the inclusive production rates.

The physical ingredients of the model are the following:

- Although the underlying theory should be ideally QCD there is a sharp distinction between soft dynamics, which provides the binding of the partons in the hadrons and also the final hadronization of the shaken-off partons, and hard dynamics that causes the parton scattering.
- Hard collisions gives a finite transverse momentum to the partons, which remains however small with respect to the typical longitudinal momentum. Hard rescattering is included, but not hard branching of the partons.
- Discrete quantum numbers like spin and colour are not taken into account.

The observable quantities that can be computed are the hard inelastic cross section, the inclusive cross section for the production of back-to-back pairs of partons, the cross section for double pair production, the multiplicity distribution, the backward forward correlation between the produced partons. The effect on the observables of the hard dynamics, described by the Hamiltonian, will be distinguishable from the effect of the hadronic structure, which is parametrized in an independent way.

In the second chapter, after a short reminder of the eikonal formalism [1], whose use is suggested by the kinematical conditions, the general features of the model are made precise by defining the interaction and by choosing a definite partonic description of the hadron. In the third chapter the expression for the inelastic cross section is derived. In the fourth chapter a few observables related to the production process are calculated. In the fifth chapter the possibility of a non uniform distribution of the partonic matter in the hadron is considered and the effects of this hypothesis on the inelastic cross section and on the production rates are worked out. Since the whole treatment deals with the transverse variables an exploration on the possibility of taking, somehow, into account the longitudinal degrees of freedom and of some simple related consequences is presented in the sixth chapter where also an initial discussion of more general forms of parton distribution inside the hadron is sketched.

One can find lot of previous treatment sharing important analogies with the treatment presented here, both in the eikonal formulations for the multiple production and in some purely probabilistic descriptions of the collision processes, which however, providing a form of unitarization of the transition probabilities, are in their final answer, analogous to the present formulation[2]. The main advantage in

the actual approach with the Hamiltonian formalism is that unitarity is explicitly implemented in all the different steps of the calculation and its role, with respect to the different observables considered, can be always traced back.

A preliminary version of this work was presented at the *XXVIII International Symposium on Multiparticle Dynamics - Delphi (1998)*[3].

2. General features

2.1 A short reminder of the eikonal approximation

The description and justification of the eikonal formalism in high energy scattering has been presented in a lot of papers, so there is no point in re-deriving it. Only some features that are relevant for the next exposition are here briefly recalled [1,4].

The relative motion of two very fast colliding particles in their c.m. frame is described by the free Hamiltonian;

$$H_o = \mathbf{v} \cdot \mathbf{p} + M/\gamma$$

One then adds an interaction term V which is for the moment left unspecified, but for the fact that it depends on the relative coordinate $\mathbf{r} = (\mathbf{B}, z)$. At very high energy the speed remains practically constant even for sizable changes in the momentum, so the solution of this Hamiltonian problem is given by the wave-function:

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \exp \left[-\frac{i}{\beta} \int_{-\infty}^z dz' V(\mathbf{B}, z') + ipz \right] \quad (2.1)$$

The exponential factor yields, in the limit $z \rightarrow \infty$, the S-matrix at fixed B . From this formula it appears clearly that the most important feature of the potential is its dependence on B whereas its longitudinal variables are always integrated; moreover the integration in dz'/β is equivalent to an integration over time.

The considerations here presented are of pure kinematical origin, they keep their validity also when a more complicated structure is foreseen both for the interaction term and for the incoming states as it will be done in the next section. The main improvement will be the description of the incoming states as systems with internal degrees of freedom [5,6] and thus the introduction of a set of transverse coordinates, one for each component.

2.2 Description of the model

The model describes the hadrons as sets of bound partons which, due to the interaction, may become finally free and eventually may be detected as jets in the final state; the hadronization process is not described.

The only detailed kinematics is the transverse one, the longitudinal is in some way integrated over, so also the longitudinal relative motion of the hadrons, which appears in eq.(2.1) as e^{ipz} , is not explicitly written out. Since we deal with very-high energy collisions there is a sharp distinction between backward and forward degrees of freedom. We call a_b, a_f the operators of the bound backward and forward partons and c_b, c_f the operators of the free partons. They have the standard commutation relations, every backward operator commutes with every forward operator and every a commutes with every c ; they are local in \mathbf{b} , the transverse impact parameter of the parton, this is possible since the size of the region relevant for the hard scattering is much smaller than the hadron size; so we can write a free Hamiltonian:

$$\mathcal{H}_o = \sum_{v=f,b} \omega \int d^2b [a_v^\dagger(\mathbf{b})a_v(\mathbf{b}) + c_v^\dagger(\mathbf{b})c_v(\mathbf{b})] \quad (2.2)$$

The interaction that we want to describe is the hard collision of two bound partons that give rise to two free partons in such a way however that they keep their property of being either backward or forward. Thus the interaction Hamiltonian is written as:

$$\begin{aligned} \mathcal{H}_I &= \lambda \int d^2b h_b(\mathbf{b})h_f(\mathbf{b}) \\ h_v(\mathbf{b}) &= c_v^\dagger(\mathbf{b})a_v(\mathbf{b}) + a_v^\dagger(\mathbf{b})c_v(\mathbf{b}) . \end{aligned} \quad (2.3)$$

With this choice the interaction and the free Hamiltonian commute: $[\mathcal{H}_o, \mathcal{H}_I] = 0$, the theory however is not trivial, even though it has been simplified, the S-matrix can be written out in the form $\mathcal{S} = \exp[-i\mathcal{H}\tau]$ where τ is an interaction time.

2.3 Discretization

The complete locality of the interaction in the transverse coordinates is both unrealistic and sometimes inconvenient, we consider an alternative with a finite size Δ of the hard interaction and a discretization of the transverse plane. The size Δ is related to the cut-off in the transverse momentum which must be put in order to be allowed to perform perturbative calculations, so the natural choice is $\Delta \approx p_\perp^{-2}$; this choice leads also to the interpretation of $\tau \approx 1/p_\perp \approx \sqrt{\Delta}$. The commutation relations become $[A_{v,j}, A_{u,i}^\dagger] = \delta_{i,j}\delta_{u,v}$, and so on. So in this discrete version the emission and absorption operators are dimensionless; also the coupling constant g is dimensionless, it is related to the previous coupling constant by $\lambda = g\sqrt{\Delta}$. The parameter Ω plays no role in the next treatment, it might also coincide with the

previously introduced ω . In this way we get

$$\begin{aligned}\mathcal{H}_o &= \sum_{v,j} \Omega [A_{v,j}^\dagger A_{v,j} + C_{v,j}^\dagger C_{v,j}] \\ \mathcal{H}_I &= (g/\sqrt{\Delta}) \sum_j H_{b,j} \cdot H_{f,j} \\ H_{v,j} &= C_{v,j}^\dagger A_{v,j} + A_{v,j}^\dagger C_{v,j}\end{aligned}\tag{2.4}$$

Also the S-matrix becomes discretized and it takes the form

$$\mathcal{S} = \prod_j \mathcal{S}_j \quad , \quad \mathcal{S}_j = \exp[-i(g/\sqrt{\Delta})\tau H_{b,j} \cdot H_{f,j}] \quad ,\tag{2.5}$$

With the previous interpretation of τ , one obtains the simpler expression:

$$\mathcal{S}_j = \exp[-ig H_{b,j} \cdot H_{f,j}] \quad .\tag{2.5'}$$

In order to apply this model one must choose a definite initial state; it will be factorized in the same way as the S-matrix: as far as its structure in a site j is concerned there are no strong indications. A possible choice, related to some theoretical ideas about the non perturbative partonic structure of the hadron[7,8], is the coherent state, so we may write

$$|I\rangle = \prod_j |I\rangle_j \quad , \quad |I\rangle_j = \exp[-\frac{1}{2}(|F_b|^2 + |F_f|^2)] \exp[F_b A_b^\dagger + F_f A_f^\dagger] | \rangle_j \tag{2.6}$$

The vacuum state is also formally factorized; it has to be noted that the weight F of the coherent state may vary from site to site. For simplicity the index j will be not written out, whenever possible.

It is useful to introduce the auxiliary operators through the definitions

$$\begin{aligned}P &= (C + A)/\sqrt{2} & Q &= (C - A)/\sqrt{2} \\ A &= (P - Q)/\sqrt{2} & C &= (P + Q)/\sqrt{2}\end{aligned}\tag{2.7}$$

in this way we get

$$\mathcal{H}_o = \sum_{v,j} \Omega [P_{v,j}^\dagger P_{v,j} + Q_{v,j}^\dagger Q_{v,j}] \quad , \quad H_{v,j} = P_{v,j}^\dagger P_{v,j} - Q_{v,j}^\dagger Q_{v,j}\tag{2.8}$$

and it is also easy to express $|I\rangle_j$ in term of the basis generated by P and Q .

3. Inelastic cross section

3.1 Contribution of a discrete site

The first observable that can be calculated within the model is the inelastic cross section; it will be calculated using the discrete formulation for the states and for the S-matrix. In the basis generated by the operators P and Q the operator \mathcal{S}_j is diagonal, so it is easy to calculate the matrix element $S_j = {}_j\langle I|\mathcal{S}_j|I\rangle_j$, it has the expression:

$$S_j = N^2 \sum_{k_1 \dots k_4} \frac{1}{k_1!k_2!k_3!k_4!} \left(\frac{1}{2}|F_b|^2\right)^{k_1+k_2} \left(\frac{1}{2}|F_f|^2\right)^{k_3+k_4} \exp[-ig(k_1 - k_2)(k_3 - k_4)] . \quad (3.1)$$

The indices k_1, k_2, k_3, k_4 refer to the quanta created respectively by P_b, Q_b, P_f, Q_f . The normalizing factor is given in eq.(2.6), actually

$$N = \exp\left[-\frac{1}{2}(|F_b|^2 + |F_f|^2)\right] . \quad (3.1')$$

By using the following representation, with $\alpha\beta = g$,

$$\exp[-ig(k_1 - k_2)(k_3 - k_4)] = (2\pi)^{-1} \int dudv \exp[iuv + i\alpha u(k_1 - k_2) + i\beta v(k_3 - k_4)]$$

the multiple sum in the expression of S_j can be transformed into an integral. We use the positions $T_v = |F_v|^2$ and we obtain:

$$S_j = \frac{1}{2\pi} \int dudv \exp[iuv] \exp[-T_b(1 - \cos \alpha u) - T_f(1 - \cos \beta v)] . \quad (3.2)$$

3.2 Continuum limit

When the distribution functions F_v do not vary strongly from site to site one can devise a continuum limit * so that the partonic structure of the colliding hadrons is described as:

$$|I\rangle = \exp \int d^2b \left[-\frac{1}{2} [|f_b(\mathbf{b})|^2 + |f_f(\mathbf{b} - \mathbf{B})|^2] + [f_b(\mathbf{b})a_b^\dagger(\mathbf{b}) + f_f(\mathbf{b} - \mathbf{B})a_f^\dagger(\mathbf{b})] \right] | \rangle .$$

Here \mathbf{B} denotes the relative impact parameter of the two hadrons. We consider the natural relation $F \approx f\sqrt{\Delta}$; therefore, if f is not singular, in the expression for S_j the factors T become small, so the exponential in the integral representation can be expanded and integrated term by term giving as a result:

$$S_j \approx 1 - T_b T_f (1 - \cos g) + \frac{1}{2} T_b T_f (T_b + T_f) (1 - \cos g)^2 + \dots \quad (3.3)$$

* This means precisely that the distributions are smooth, not that we consider the unrealistic limit $p_\perp \rightarrow \infty$

With our normalization the inelastic cross section at fixed hadronic impact parameter B is

$$\sigma(\mathbf{B}) = 2 < I | (1 - \Re \mathcal{S}) | I > - | < I | (1 - \mathcal{S}) | I > |^2 . \quad (3.4)$$

The product of the matrix elements $\mathbf{S} = \prod_j S_j$ is, as usual, calculated through the sum of the logarithms and the sum $\Delta \sum_j$ is finally converted into the integration $\int d^2b$. The final result is:

$$\sigma(\mathbf{B}) = 1 - \exp \int d^2b \left[-\hat{\sigma} t_b(\mathbf{b}) t_f(\mathbf{b} - \mathbf{B}) + \right. \\ \left. \frac{1}{4} \hat{\sigma}^2 t_b(\mathbf{b}) t_f(\mathbf{b} - \mathbf{B}) (t_b(\mathbf{b}) + t_f(\mathbf{b} - \mathbf{B})) + \dots \right] . \quad (3.5)$$

Two definition are introduced, $\hat{\sigma} = 2\Delta(1 - \cos g)$, since this is precisely the parameter which has the role of elementary partonic cross section, and $t_v(\mathbf{b}) = |f_v(\mathbf{b})|^2$, giving the transverse density of bound partons.

The form of the inelastic cross section is quite usual. The second term in the argument of the exponential represents the rescattering corrections, which will be discussed below, where a nonuniform model of the hadron will be explored. The cross section arises from the integration over the impact parameter, so the result depends to a large extent on the properties of the density functions t_v .

As a simplest case the distribution can be taken to be completely uniform in b :

$$t_b(\mathbf{b}) = \rho_b \vartheta(R - |b|) \quad , \quad t_f(\mathbf{b}) = \rho_f \vartheta(R - |b|) ; \quad (3.6)$$

elementary geometrical considerations give $|B| = 2R \cos \frac{1}{2}\gamma$ with $0 \leq \gamma \leq \pi$. The exponent in the integrand is given by the partial superposition of the two disks, the superposition area is

$$W = R^2(\gamma - \sin \gamma) . \quad (3.7)$$

and the cross section is expressed as:

$$\sigma_{in} = 2\pi R^2 \int_0^\pi d\gamma \sin \gamma \left[1 - \exp[-\nu \xi] \right] , \quad (3.8)$$

where $\nu = \hat{\sigma} \rho_b \rho_f \pi R^2$ and $\xi = (\gamma - \sin \gamma)/\pi = W/(\pi R^2)$.

With the previous interpretation of $\hat{\sigma}$, the numerical constant $\nu \xi$ is the mean number of partonic interactions. In the limit of wholly uniform density the introduction of corrective terms like those appearing in the expression of S_j , eq. (3.3), amounts simply to a redefinition of ν .

It is possible to give a simple analytical form for σ_{in} in the two limiting situations of very small or very large ν . In the first case one gets

$$\sigma_{in} = \pi R^2 \cdot \nu . \quad (3.8')$$

In the second case we can start from the expression

$$\sigma_{in} = 2\pi R^2[2 - D(\nu)] \quad (3.8'')$$

where the real function $D(\nu)$ defined by eq(3.8) is monotonically decreasing, for large ν it results $D(\nu) \approx 2(6\nu^2/\pi^2)^{-1/3} \cdot \Gamma(\frac{2}{3})$ so that the geometrical limit of black disks $4\pi R^2$ is approached.

4. Inclusive cross sections and multiplicity distribution

4.1 Pair and double-pair production

The production of a pair is signaled both by the production of a backward parton and by the production of a forward parton, the Hamiltonian being fully symmetric. We make then the arbitrary choice of looking at the forward particles only; successively we shall investigate how much the rescattering processes may destroy the sharp correlation between backward and forward scattered partons. We start from the computation of the inclusive production from a single site *

$$\langle X_j \rangle = {}_j \langle I | \mathcal{S}_j^\dagger C_f^\dagger C_f \mathcal{S}_j | I \rangle_j \quad (4.1)$$

and we observe that H_b can be treated as a number with respect to the forward operators, so the following relation holds:

$$\mathcal{S}_j^\dagger C_f \mathcal{S}_j = C_f \cos(gH_b) + iA_f \sin(gH_b) . \quad (4.2)$$

The state $|I \rangle_j$ is a coherent state of bound partons and contains no free parton so one obtains:

$$\langle X_j \rangle = {}_j \langle I | A_f^\dagger A_f \sin^2(gH_b) | I \rangle_j . \quad (4.1')$$

The matrix element in the previous expression is better calculated in the basis generated by the operators P and Q (details are given in the Appendix), and the result is

$$\langle X_j \rangle = \frac{1}{2} T_f [1 - \exp[-T_b(1 - \cos(2g))]] . \quad (4.3)$$

We can now go to the continuum limit, always under the hypothesis of smooth distributions $t_v(\mathbf{b})$; we find a problem in the presence of the function $\cos(2g)$ instead of $\cos g$, in fact the elementary partonic interaction enters in $\langle X_j \rangle$ through the

* The production of a double pair in a single site is not included because the size Δ is defined by the hard interaction, so the production of four particle in a single site would not give rise, if not accidentally, to two distinct pairs, each with compensating momenta.

quantity $\kappa = \frac{1}{2}\Delta[(1 - \cos(2g))]$, which is related to $\hat{\sigma}$ in this way: $\kappa = \hat{\sigma} \cdot \frac{1}{2}[1 + \cos g]$. The two constants coincide for small g , where both take the same value: $\hat{\sigma} = \kappa = g^2\Delta$, higher powers in g^2 make however them different. Unitarity corrections at the level of parton-parton collisions are in fact different in the total and in the inclusive cross section.

In the simple case of small values of T we expand the exponential of eq. (4.3) and we get the usual expression:

$$X(\mathbf{B}) = \kappa \int d^2b t_f(\mathbf{b}) t_b(\mathbf{b} - \mathbf{B}) \quad (4.4)$$

and when we consider the integration over the hadronic impact parameter

$$D_1 = \int X(\mathbf{B}) d^2B \quad (4.5)$$

the two integrals get factorized. When, however, in the same conditions we calculate the double pair production we do not end with a factorized expression (there are four factors and three integrations). In this case we obtain

$$D_2 = \kappa^2 \int d^2B d^2b d^2b' t_f(\mathbf{b}) t_b(\mathbf{b} - \mathbf{B}) t_f(\mathbf{b}') t_b(\mathbf{b}' - \mathbf{B}); . \quad (4.6)$$

In the limit of rigid disk, using the geometrical considerations and the definition of the previous section 3.2 the expression becomes:

$$D_2 = 2(\pi R^2)^3 (\rho_b \rho_f \kappa^2)^2 \int \xi^2 \sin \gamma d\gamma = (\pi R^2)^3 (\rho_b \rho_f \kappa)^2 [1 - 16/(3\pi^2)] . \quad (4.7)$$

A ratio of the quantities that have been now calculated and that is of phenomenological interest is $\sigma_{\text{eff}} = [D_1]^2/D_2$, [9,10] which has the nice property, within this treatment, of being independent of κ . It depends rather on the space behavior of $t(\mathbf{b})$, namely on the hadron shape. Since in the rigid disk limit it results from eq (4.4,5) that

$$D_1 = (\pi R^2) \kappa \rho_f \rho_b \quad (4.5')$$

the ratio we are looking for is given by:

$$\sigma_{\text{eff}} = \frac{\pi R^2}{1 - 16/(3\pi^2)} \approx 2.2\pi R^2 , \quad (4.8)$$

The previously calculated expression of σ_{in} is really the hard part of the total inelastic cross section, where as "hard" part we mean the contribution of all the events with at least one hard scattering. If we believe that, in going on with the total energy these events become dominating we would like to have this term not

too small with respect to the experimental σ_{in} , which in turn appears to be sizably larger, of about a factor 2, with respect to σ_{eff} , so in this model we expect to approach at high energy the black-hadron limit which produces $\sigma_{in} = 4\pi R^2$.

4.2 Multiplicity distribution

The distribution of the multiplicities of the produced pairs is calculated by defining the projection operator over the number of free partons. Since there is a sharp distinction between backward and forward particles we can choose to take the forward parton as a signal of the pair production. For a fixed site j the number projector is *

$$\mathcal{P}_n = \frac{1}{n!} : C^\dagger C e^{-C^\dagger C} : \quad (4.9)$$

The colon indicates the normal ordering of the C -operators, more precisely if the operators refer to the forward particles this is the forward normal ordering, and in no way it affects the backward operators. The properties of the \mathcal{P}_n operators are easily verified. They are evidently diagonal in the number basis and clearly $\langle m | \mathcal{P}_n | m \rangle = 0$ when $m < n$, for $n \leq m = n + \ell$ we get, through direct computation

$$\langle m | \mathcal{P}_n | m \rangle = \frac{1}{n!} \sum_{k=0}^{\ell} (-)^k \frac{(n + \ell)!}{(\ell - k)! k!} = \frac{(n + \ell)!}{\ell! n!} [1 - 1]^\ell = \delta_{\ell,0} .$$

It could be more convenient to deal with the generator of the projectors:

$$\mathcal{P}_n = \frac{1}{n!} \left(\frac{\partial}{\partial \mu} \right)^n \mathcal{Z} \Big|_{\mu=-1} \quad \text{with} \quad \mathcal{Z} = : e^{\mu C^\dagger C} : . \quad (4.10)$$

An auxiliary function is introduced

$$Z(\mu) = \langle I | \mathcal{S}^\dagger \mathcal{Z} \mathcal{S} | I \rangle \quad (4.11)$$

Calculations are strongly simplified by the normal ordering; through the same steps which lead from eq.(4.1) to eq.(4.3) one can get in fact

$$Z(\mu) = \exp\left[\frac{1}{2}\mu T_f\right] \langle I | \exp\left[-\frac{1}{2}\mu T_f \cos(2gH_b)\right] | I \rangle \quad (4.12)$$

and one must remember that the functions of H_b are *not* normal ordered. The matrix element of eq (4.12) is computed by expanding the operator into numerical Bessel functions and operatorial trigonometric functions, using the relation[11]: $e^{z \cos \theta} =$

* Whenever possible the indices f, b, j are suppressed

$I_o(z) + 2 \sum_k I_k(z) \cos(2k\theta)$, which allows to perform the same calculations leading from eq.(4.1) to eq.(4.3). The final expression is written as

$$Z(\mu) = \exp[\tfrac{1}{2}\mu T_f] \left[I_o(-\tfrac{1}{2}\mu T_f) + 2 \sum_k I_k(-\tfrac{1}{2}\mu T_f) \exp[-T_b(1 - \cos(2kg))] \right] . \quad (4.12')$$

According to eq (4.10) the derived multiplicity distribution, as seen from a single site is

$$K_n = \frac{1}{n!} \left(\frac{\partial}{\partial \mu} \right)^n Z(\mu) \Big|_{\mu=-1} .$$

When the production at the single site is not very strong it is natural to expand in the absorbing term T_b . A straightforward although a bit lengthy calculation yields, to the second order in T_b :

$$K_n = \frac{1}{n!} [\tfrac{1}{2}T_f]^n \left[T_b(1 - T_b)[1 - \cos(2g)]^n \exp\left(-\tfrac{1}{2}T_f[1 - \cos(2g)]\right) + \tfrac{1}{4}T_b^2[1 - \cos(4g)]^n \exp\left(-\tfrac{1}{2}T_f[1 - \cos(4g)]\right) \right] + \dots \quad (4.13)$$

The result is evidently a sum of two Poissonian distributions. The origin is the Poissonian distribution of the initial coherent state, which is modified in a well defined way by re-interactions.

When re-interactions are important there is no simple expression for the result, as usually we get a not too awkward expression in the extreme limit *i.e.*, very large T_b . In this case we could neglect in eq.(4.12') all the terms containing the negative exponent of T_b . We translate[11] afterwards the expression containing the Bessel function into one expressed through the confluent hypergeometric function:

$$Z(\mu) \approx M(\tfrac{1}{2}; 1; \mu T_f) ,$$

and we obtain in this way for the multiplicity distribution the form:

$$K_n \approx \frac{1}{n!} T_f^n \exp[-T_f] \cdot L_n \quad , \quad L_n = \frac{1}{n!} \left(\tfrac{1}{2} \right)_n M(\tfrac{1}{2}; n+1; T_f) \quad (4.14) .$$

The expression has been put into a form of a Poissonian distribution times another factor, this further factor L_n is not a small correction however, it changes in an essential way the shape of the distribution.

4.3 Forward-backward correlations

Until now pair production has been described by looking at the production of a definite component of the pair. When the re-interactions are not very important this attitude is justified. If however there is a strong localized production in one site further investigations are needed. In this last case we can calculate within the model the variance and the covariance of the number of emitted partons. We start from the computation of the dispersion in the inclusive production in a single site j :

$$\langle X^2 \rangle = \langle I | \mathcal{S}^\dagger C_f^\dagger C_f C_f^\dagger C_f \mathcal{S} | I \rangle . \quad (4.15)$$

The calculation goes along the same patterns as in the previous calculation of $\langle X \rangle$. The result is

$$\langle X^2 \rangle = T_f^2 \langle I | \sin^4(gH_b) | I \rangle + T_f \langle I | \sin^2(gH_b) | I \rangle$$

and for the variance one obtains:

$$\begin{aligned} \Sigma_f = \langle X^2 \rangle - \langle X \rangle^2 = & \frac{1}{2} T_f [1 - \exp[-T_b(1 - \cos(2g))]] + \frac{1}{8} T_f^2 \times \\ & \left(1 - 2 \exp[-2T_f[1 - \cos(2g)]] + \exp[-T_f[1 - \cos(4g)]] \right) . \end{aligned} \quad (4.16)$$

The starting point to calculate the covariance is

$$\langle W \rangle = \langle I | \mathcal{S}^\dagger C_f^\dagger C_f C_b^\dagger C_b \mathcal{S} | I \rangle , \quad (4.17)$$

Because of the coherent state structure of $|I\rangle$ one can show that

$$\mathcal{S}^\dagger C_b \mathcal{S} | I \rangle = i F_b \sin(gH_f) | I \rangle .$$

By using this relation the expression of $\langle W \rangle$ gets simplified to some extent. With some more work one recognizes also that

$$\begin{aligned} A_f \sin(gH_f) | I \rangle &= F_f \cos g \sin(gH_f) | I \rangle \\ C_f \sin(gH_f) | I \rangle &= F_f \sin g \cos(gH_f) | I \rangle \end{aligned}$$

From now on the calculations uses the results already seen, like eq .(4.1,1',3) and yields for the covariance $\Sigma_{f,b} = \langle W \rangle - \langle X_f \rangle \langle X_b \rangle$ the following expression:

$$\Sigma_{f,b} = \frac{1}{2} T_f T_b \sin^2 g \left[\exp[-T_b\{1 - \cos(2g)\}] + \exp[-T_f\{1 - \cos(2g)\}] \right] \quad (4.18)$$

In term of the quantities $\Sigma_{f,b}$ one can define the correlation coefficient as:

$$\rho_{f,b} = \frac{\Sigma_{f,b}}{[\Sigma_f \Sigma_b]^{1/2}} \quad (4.19)$$

A look to eq (4.16,18,19) shows, as expected, that for small values of T_v , namely for small production, the correlation goes to 1, on the contrary when T_v becomes large the correlation goes to zero.

5. Non uniform hadrons

5.1 Inelastic cross section

We wish now to explore the possibility that the hadron, and its projection over the transverse plane, shows strong inhomogeneities in the matter density. This is represented by assuming the existence of black spots, that cover a limited amount of the transverse area, while a much fainter "gray" background smoothly fills the rest of the hadron. For the black spots the continuous limit (sect. 3.2) is not justified, but since they cover globally a small area, we can distinguish three possible kinds of collisions: The spot-spot collision, to be treated individually, the spot-background collision, where the background is treated as a continuum in b and finally the background-background which is nothing but the continuous limit already studied. In order to deal with the first case we start again from eq. (3.2), in the limit of very large T_j the integral will be expanded around the zeros of the exponent by setting:

$$u\alpha = 2\pi n + \chi \quad , \quad v\beta = 2\pi m + \phi ;$$

the subsequent Gaussian integrations over χ and ϕ give:

$$S_j \approx \frac{1}{g} \frac{1}{\sqrt{T_f T_b}} \sum_{m,n} \exp[i(2\pi)^2 mn/g] \quad (5.1)$$

In order to give an estimate of its value, the double sum is then converted into a double integration from $-\infty$ to $+\infty$ with the final result:

$$S_j \approx S_s = \frac{1}{2\pi} \frac{1}{\sqrt{T_f T_b}} \quad (5.2)$$

In the mixed spot-background collision, always taking eq (3.2) as starting point, it is possible to expand in one of the T_v terms, keeping the full expression for the other one with the results:

$$\begin{aligned} S_j \approx S_f &= 1 - T_b(1 - \exp[-T_f(1 - \cos g)]) & \text{or} \\ S_j \approx S_b &= 1 - T_f(1 - \exp[-T_b(1 - \cos g)]) \end{aligned} \quad (5.3)$$

for the collision between a forward spot with the backward background and between a backward spot and the forward background. The background-background contribution has been already given in eq.(3.3).

We are now in position to calculate the modification to the expression for the inelastic cross section, eq.(3.8), that are introduced by the hypothesis of the existence of the black spots. The gray distribution is again assumed to be uniform so the term in eq.(3.3) is constant and it will be denoted by S_o . The interaction area W is considered as composed by w small elements $W = w\Delta$, the ratio $\xi = W/\pi R^2$, will also be frequently used, so that $w = \xi(\pi R^2/\Delta)$.

The quantum mechanical treatment of a nonuniform hadron is now presented in a form which contains certainly some rough simplifications that were unavoidable in order to deal with a system with an elevated degree of complexity. We may begin by assuming, at fixed j , a state of the kind: $|I >_j = |I_b >_j |I_f >_j$

$$|I_b >_j = \left[x \exp[-\frac{1}{2}|F_b|^2] \exp[F_b A_b^\dagger] + y \exp[-\frac{1}{2}|G_b|^2] \exp[G_b A_b^\dagger] \right] | > \quad (5.4)$$

In this expression, and in the similar one for $|I_f >$, the terms $F = |F|e^{i\phi}$ and $G = |G|e^{i\chi}$ denote two different thickness while the coefficients of the q.m. superposition are x, y . Since the two coherent states are not orthogonal the general form of the normalization condition is complicated:

$$1 = |x|^2 + |y|^2 + \exp[-\frac{1}{2}(|F| - |G|)^2] \{ xy^* \exp[|FG|[1 - e^{i(\phi-\chi)}]] + x^* y \exp[|FG|[1 - e^{i(\chi-\phi)}]] \}$$

When however the two thickness are very different the last term in the normalization condition is exponentially depressed and we are left with

$$|x|^2 + |y|^2 \approx 1 .$$

It is reasonable to expect that the same feature occurs also in calculating the relevant matrix elements, but in a significant case the calculations will be carried out explicitly so that the guess can be verified. In fact a non diagonal term of the matrix element of \mathcal{S}_j is computed along the same lines yielding eq.(3.1,2) and the result is:

$$S_{ND} = (2\pi)^{-1} \int dudv \exp[iuv] \exp[-\frac{1}{2}(|F_b| - |G_b|)^2 - \frac{1}{2}(|F_f| - |G_f|)^2] \times \exp[-|F_b G_b|(1 - e^{i(\phi-\chi)_b} \cos \alpha u)] \exp[-|F_f G_f|(1 - e^{i(\phi-\chi)_f} \cos \beta v)] \quad (5.5)$$

It is therefore enough to have one F very different from one G in order to obtain an exponentially suppressed contribution.

One can see in this way that out of the 16 terms, that are produced in calculating the complete matrix element of \mathcal{S}_j , 12 *i.e.* the interference terms, are exponentially suppressed.

It is useful to formalize this approximate result in the following way: let be $|I_b >_j = [|\circ > + \mu|\bullet >]_b$ where

$$\langle \circ|\bullet \rangle \approx \langle \bullet|\circ \rangle \approx 0 \quad \text{and} \quad \langle \circ|\circ \rangle \approx \langle \bullet|\bullet \rangle \approx 1 \quad (5.6)$$

Then ${}_j \langle I_b | I_b \rangle_j \approx 1 + \mu^2$ and the S-matrix element takes the form*

$$S_j = [\langle \circ | + \mu \langle \bullet |]_{b,j} [\langle \circ | + \mu \langle \bullet |]_{f,j} \mathcal{S}_j [|\circ \rangle + \mu |\bullet \rangle]_{b,j} [|\circ \rangle + \mu |\bullet \rangle]_{f,j} (1 + \mu^2)^{-2}$$

which more explicitly gives

$$S_j = [S_o + \mu^2 S_b + \mu^2 S_f + \mu^4 S_s] (1 + \mu^2)^{-2}. \quad (5.7)$$

In the limit of totally black spots, where we have the following values for the terms entering in eq.(5.7), see eq.s (3.3,3.6,5.3,5.2)

$$\begin{aligned} S_o &= 1 - \hat{\sigma} \rho_b \rho_f \Delta \\ S_b &= 1 - \rho_f \Delta = S_o \eta_b \\ S_f &= 1 - \rho_b \Delta = S_o \eta_f \\ S_s &= 0 \end{aligned} \quad (5.8)$$

the expression for the S-matrix is:

$$\begin{aligned} \mathbf{S} &= \prod_j S_j = S_o^w \cdot [1 + \mu^2 \eta_b + \mu^2 \eta_f]^w (1 + \mu^2)^{-2w} \\ &= \mathbf{S}_o \mathbf{S}_c \end{aligned} \quad (5.9)$$

where the factor $\mathbf{S}_o \equiv S_o^w$ gives the contribution of the background scattering.

A much simpler expression can be written when μ is small and keeping into account that \mathbf{S}_o is not very different from 1:

$$\mathbf{S}_c = 1 - w \mu^2 \Delta [\rho_b (1 - \hat{\sigma} \rho_f) + \rho_f (1 - \hat{\sigma} \rho_b)] \quad (5.10)$$

One may notice that the first factor, $w \mu^2 \Delta$, represents the part covered by black spots within the interacting area of the hadrons at given \mathbf{B} . A more general observation is that the systematic use of the relations of eq.(5.5,6) eliminates the more complicated aspects induced by quantum mechanics and the answer is the same as if one would have taken a probabilistic distribution of spots in the transverse section of the interacting hadrons.

5.2 Pair and double pair production

The basic ingredient for these new calculations is always given by the expression in eq. (4.3), which must be particularized for the situations under examinations.

* The parameter μ measures the ratio of the area covered by the spots to the background; it is assumed small and real because the relative phases have no role

For simplicity and, more, in order to put in evidence the features of this particular model the background will be taken as thin so that in eq. (4.3) the first term of the expansion of the exponential is enough while in the case of the spots the exponential will be considered totally absorbing. The local production amplitude is the sum of 4 terms, it will be indicated as

$$\langle X_j \rangle = [X_{o,o} + \mu^2 X_{s,o} + \mu^2 X_{o,s} + \mu^4 X_{s,s}](1 + \mu^2)^{-2} \quad (5.11) .$$

The first term represents the pure background interaction, it is given by eq.(4.4), the second term represents a forward spot interacting with the backward background so T_f is large while $T_b = t_b \Delta$, the third term represents the opposite situations, note that the resulting expression is *not* symmetrical because we are looking to the forward produced particles, finally the fourth term gives the effect of the spot-spot interaction. The four terms must be summed over the allowed values of j , *i.e.* over the position included in the interaction region. The sums may be expressed as :

$$\begin{aligned} \sum_j X_{o,o} &= \kappa \rho_f \rho_b W \\ \sum_j X_{s,o} &= \kappa \rho_b (T_f / \Delta) W \\ \sum_j X_{o,s} &= \frac{1}{2} \rho_f W \\ \sum_j X_{s,s} &= \frac{1}{2} (T_f / \Delta) W . \end{aligned} \quad (5.12)$$

The dependence on the angular variable γ is the same for the four terms, *i.e.* we can in any case write $\sum X = K\xi$. The integration over the impact parameter is factorized, so we get for the inclusive production rates expressions as in eq.s (4.5',7). Actually

$$D_1 = \pi R^2 K \quad , \quad D_2 = \pi R^2 K^2 [1 - 16/(3\pi^2)]$$

The result for σ_{eff} is therefore precisely the same as in eq. (4.8). It has to be noticed that although T_f may be large (T_f/Δ) in finite and would remain finite even in a formal limit $\Delta \rightarrow 0$. At first sight it seems that nothing is gained by introducing an inhomogeneity into the hadron, but in fact some new features are present. The expression of σ_{eff} is purely geometrical, it does not contains μ , R is simply the radius of the area where the spots may be found; the expression of σ_{in} on the contrary contains dynamical parameters. It is therefore instructive to compare σ_{in} and σ_{eff} .

A clear, although unrealistic example is the limiting case in which the background is so thin that it contributes negligibly to the inelastic cross section, then the S-matrix element, depends only on the spot-spot interaction and it has the form

$$\mathbf{S} = \left[1 - \frac{\mu^4 (1 - S_s)}{(1 + \mu^2)^2} \right]^w \quad (5.13)$$

For small values of μ this expression may approach 1 and so the inelastic cross section becomes small, this situation would correspond to have few spots wholly black distributed in a wide and very thin background. The conclusion of this analysis is then that in order to have $\sigma_{\text{eff}} < \sigma_{\text{in}}$ the hadron should be compact, *i.e.* without holes or transparent regions. There is however another possibility. The spots can in fact be distributed with some correlation among themselves, this possibility can be investigated, to some extent, within the model. The analysis is however formally awkward and not very conclusive, so it will be just mentioned but not reported explicitly.

5.3 Local effects of unitarity

Until now the models of the hadron that have been considered in more detail cover two extreme situations, the case where there is not local rescattering and unitarity is relevant only to the whole hadronic interaction, allowing a description of the multiple disconnected partonic collisions, and the case where the re-interaction is so strong that, locally, the hadron is completely absorbing. We think useful to investigate briefly some intermediate situation, in this context sometimes the matter distribution in the hadron (the bound-parton distribution of the model) will be assumed to have a Gaussian shape, just in order to allow some explicit analytic calculation.

We start from the inclusive productions: for the single inclusive we have in general in the model, comparing eq.s (4.3,4,5)

$$D_1 = \frac{1}{2} \int t_f(\mathbf{b} + \mathbf{B}) \left[1 - \exp[-2\kappa t_b(\mathbf{b})] \right] d^2 b d^2 B \quad (5.14)$$

and for the double inclusive it results

$$D_2 = \frac{1}{4} \int t_f(\mathbf{b} + \mathbf{B}) t_f(\mathbf{b}' + \mathbf{B}) \left[1 - \exp[-2\kappa t_b(\mathbf{b})] \right] \left[1 - \exp[-2\kappa t_b(\mathbf{b}')] \right] d^2 b d^2 b' d^2 B \quad (5.15)$$

The densities t vary with the total energy of the process because, really, only the part of the partonic spectrum that can give rise to the hard scattering enters in eq.s(5.14,15) and this part certainly grows with the total energy of the collision. The simplest way in which this variation can be implemented is by rescaling the densities by a factor λ growing with the energy. It is evident that in calculating $\sigma_{\text{eff}} = [D_1]^2/D_2$ the factors affecting t_f are eliminated in the ratio and the overall effect of λ amounts formally to a rescaling of κ .

So, until $2\kappa t(\mathbf{b}) \ll 1$ one expands the exponential and gets the expression already displayed which contains only geometrical elements, but when $2\kappa t(\mathbf{b}) \approx 1$ the final expression is less simple. It will be studied with the particular choice

$t(\mathbf{b}) = (\mu/\pi) \exp[-\mu \mathbf{b}^2]$ and with the definitions $\mathbf{b} = \mathbf{R} + \mathbf{r}/2$ $\mathbf{b}' = \mathbf{R} - \mathbf{r}/2$, so that

$$\int t_f(\mathbf{b} + \mathbf{B}) t_f(\mathbf{b}' + \mathbf{B}) d^2 B = (\mu/2\pi) \exp[-\frac{1}{2}\mu \mathbf{r}^2] = \mathcal{F}(\mathbf{r})$$

and

$$D_2 = \frac{1}{4} \int \mathcal{F}(\mathbf{r}) \left[1 - \exp[-2\kappa t_b(\mathbf{R} + \mathbf{r}/2)] \right] \left[1 - \exp[-2\kappa t'_b(\mathbf{R} - \mathbf{r}/2)] \right] d^2 R d^2 r$$

When $2\kappa t(\mathbf{b}) < \tau$ and τ is not too small, the integral can be separated into two parts, the first is approximately

$$\int \mathcal{F}(\mathbf{r}) [1 - e^{-\tau}]^2 d^2 R d^2 r$$

with the bounds $b < b_o$, $b' < b_o$, $b_o^2 = \ln(2\mu\kappa/\pi\tau)/\mu$. The integration region in $d^2 R$ has the same geometrical shape as the integration region yielding eq (3.7), only the role of the radius is played by b_o . The integration in $d^2 r$ is always convergent and has no significant dependence on b_o , so the conclusion is that this first addendum is proportional to b_o^2 and so to $\ln \kappa$. The second addendum is certainly not growing with κ , at fixed τ , so at the end we get the result $D_2 \propto b_o^2 \propto \ln \kappa$. The behavior of D_1 is much simpler to estimate, the double integral is factorized, the first factor does not depend on κ , the second is proportional to $\ln \kappa$, for large κ as it may be seen by explicit calculation. The conclusion of the analysis shows that the ratio giving σ_{eff} has a region where it stays essentially constant also if the hadron has no sharp boundaries but that at the end it will suffer a logarithmic increase whose origin has to be found in the local (*i.e.* at fixed \mathbf{b}) unitarity. The moment at which these local unitarity effects begin to be relevant is when $2\kappa t(\mathbf{b})$ is not too small with respect to 1. It must be remarked that the growth with energy is basically different for σ_{in} and for σ_{eff} since in the first case a logarithmic expansion is given already by the simple formula, which is "unitary" at hadronic level but not yet at partonic level, whereas in the second case the partonic unitarity is essential. The logarithmic growth is due to the particular choice of a Gaussian shape, an exponential shape is required to give rise to the square-logarithmic growth.

6. On the longitudinal dynamics

In the frame of the eikonal representation that has been used throughout the paper the longitudinal variables for the partons are in general expressed by means of the fraction of the longitudinal momentum carried by the partons. This is the natural choice for the bound partons, for a free parton the usual variable is the rapidity whose connection with the fractional momentum requires the introduction

of the transverse motion. Since we are always interested in situations where the partons are tied in a clear way to the initial hadrons we shall use the fractional momentum x in every case, with the sharp distinction in forward and backward partons.*

Starting from the formulation which is discrete in the transverse variables it is possible to introduce operators which depend on the fractional momentum and give the usual commutation relations like $[A_{v,j}(x), A_{u,i}^\dagger(x')] = \delta_{i,j} \delta_{u,v} \delta(x - x')$, and so on. and write the new forms of the Hamiltonians:

$$\begin{aligned}\mathcal{H}_o &= \sum_{v,j} \int dx \Pi_v x [A_{v,j}^\dagger(x) A_{v,j}(x) + C_{v,j}^\dagger(x) C_{v,j}(x)] \\ \mathcal{H}_I &= (g/\sqrt{\Delta}) \sum_j H_{b,j} \cdot H_{f,j} \\ H_{v,j} &= \int dx [C_{v,j}^\dagger(x) A_{v,j}(x) + A_{v,j}^\dagger(x) C_{v,j}(x)]\end{aligned}\quad . \quad (6.1)$$

The hadron momenta are denoted by Π_v and the fractions x refer to the forward and backward total momenta according to the operator where they appear. In the same way the incoming states are described in terms of some distributions of longitudinal partons, the direct generalization of eq. (2.6), where the transverse factorization is maintained for the hadronic state $|I\rangle = \prod_j |I\rangle_j$ and for the vacuum:

$$|I\rangle_j = \exp\left[-\frac{1}{2} \int dx (|F_b(x)|^2 + |F_f(x)|^2)\right] + \int dx [F_b(x) A_b^\dagger(x) + F_f(x) A_f^\dagger(x)] | \rangle_j . \quad (6.2)$$

With this particular choice of the interaction term everything proceeds as before because it is again possible to define the auxiliary operators $P(x)$, $Q(x)$. If we exclude black spots and go to the continuum limit the answer is:

$$\sigma(\mathbf{B}) = 1 - \exp\left[-\hat{\sigma} \int d^2b dx t_b(\mathbf{b}, x) t_f(\mathbf{b} - \mathbf{B}, x) + \frac{1}{2} \hat{\sigma}^2 \dots\right] . \quad (6.3)$$

At this point, having put into the game the longitudinal variables it is clear that the choice $t_v = \text{const}$ finds no justification whatsoever; the most uniform choice corresponds to a uniform sphere, *i.e.* $t_v(\mathbf{b}, x) = \rho_v(x) \sqrt{R^2 - b^2} \vartheta(R - |b|)$. The role played before by the partial superposition of two disks, described by $W = \pi R^2 \xi$ is now played by the product of the two volumes that are superimposed

$$U = \int \sqrt{R^2 - \mathbf{b}^2} \sqrt{R^2 - (\mathbf{B} - \mathbf{b})^2} \vartheta(R - |b|) \vartheta(R - |B - b|) d^2b . \quad (6.4)$$

* With this limitations it is not possible to introduce the hard production into the parton dynamics. It is possible to take into account re-interactions among the produced partons; in so doing the S-matrix element acquires a further phase.

This term can be written in a variety of alternative forms, but not completely by means of usual functions, a representation that turns out to be useful is:

$$U = R^4 \int e^{i\lambda \cdot \mathbf{B}} [j_1(\lambda R)/\lambda]^2 d^2\lambda = 2\pi R^4 \int J_o(\lambda B) j_1(\lambda R)^2 d\lambda/\lambda. \quad (6.4')$$

The spherical Bessel function has the form $j_1(x) = x^{-2} \sin x - x^{-1} \cos x$. The term U appears in the definition of the inelastic cross section and gives rise to a very cumbersome expression, a much simpler form is found in the definition of the inclusive production and of the double inclusive production. The first term can be obtained also by direct integration:

$$D_1 = (4\pi R^3/3)^2 \kappa \int \rho_b(x) dx \int \rho_f(x) dx. \quad (6.5)$$

For the double pair production, the expression is:

$$D_2 = (4\pi R^3)^3 2R \left[\kappa \int \rho_b(x) dx \int \rho_f(x) dx \right]^2 \int_o^\infty [j_1(t)]^4 dt/t^3. \quad (6.6)$$

The last integral gives:

$$\mathcal{K} = \int_o^\infty [j_1(t)]^4 dt/t^3 = 0.01433$$

So we finally obtain:

$$\sigma_{\text{eff}} = \frac{2\pi R^2}{81\mathcal{K}} = 2\pi R^2/1.16 \quad (6.7)$$

7. Conclusions

The model presented and worked out in some detail allows a systematization of different aspects of the hard processes in multiparticle production and suggests also some interpretations in terms of hadron structure. The connection with QCD is not direct as it appears from the fact that the interaction term is quartic while the fundamental QCD interaction term is cubic. The reason is that in the actual approach the fundamental input is the parton hard collision, not the branching process.

All unitarity corrections for the different processes are fully explicit, even though the final analytical expressions are given for the two extreme situations: weak perturbative corrections, very strong absorption. The analysis of the effects of inhomogeneities has been carried out purposely without specifying the possible

origin of this supposed property, in fact the quantities which have been considered depend on a global characteristic, the total amount of black area, and not on other details. The density distribution in the hadron is however relevant when σ_{in} and σ_{eff} are compared. The representation of the hadron transverse area as a collection of elements of finite but much smaller extension requires a limitation of the momentum transfer which cannot go below, say, of 5GeV so that the interaction area turn out of the order of $(0.04 \text{ fm})^2$, small enough in comparison with the hadron extension. The assumption that the partons of the hadron build up, locally in \mathbf{b} , a coherent state is a theoretical prejudice. The test is difficult since the production process generally alters the multiplicity distribution, so that what finally one sees only a combination of hard and soft dynamics. To gain some better insight into the problem we have discussed, at the end of the 4th paragraph, the modification to the multiplicity distribution induced by hard rescatterings,

Further questions can be raised: one is the possible coherence or correlation at different values of \mathbf{b} , what could, at the end, also involve the role of the colour variables: in fact the transverse size of the hadron fraction over which the matrix element of \mathcal{S} has been calculated is determined by the transverse momentum transfer, which is a quantity relevant for the perturbative treatment. Larger sizes of transverse coherence may however show up in the multiple-production processes. Another question is why the incoming states should be locally coherent. More general initial states could be considered, but the new parameters one introduces in this way are at present out of control.

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Appendix

In this appendix we give some details on the calculation leading from eq (4.2) to eq (4.3) since this result is repeatedly used and it yields also a model for other similar calculations. Since the calculation is performed at fixed site, the index j will be omitted. From eq (4.1'), letting the operators A_f act on the states $|I_f\rangle$, it results furthermore.

$$\langle X_j \rangle = T_f \langle I_b | \sin^2(gH_b) | I_b \rangle . \quad (A.1)$$

Going to the basis generated by the P, Q operators it results also

$$\langle I_b | \sin^2(gH_b) | I_b \rangle = \frac{1}{2} - \frac{1}{4} \langle I_b | \exp[2ig(P_b^\dagger P_b - Q_b^\dagger Q_b)] | I_b \rangle + c.c. .$$

The state is now expanded

$$|I_b\rangle = N \sum_{m,n} \frac{1}{2^{(m+n)/2} m! n!} F_b^{m+n} (P_b^\dagger)^m (-Q_b^\dagger)^n | \rangle$$

and the exponential acts now trivially on the Fock states so the final outcome is

$$\langle I_b | \sin^2(gH_b) | I_b \rangle = \frac{1}{2} - \frac{1}{4} N^2 \sum_{m,n} \frac{1}{2^{(m+n)/2} m! n!} |F_b^2|^{m+n} \exp[2ig(m-n)] + c.c. \quad (A.2)$$

A convenient way of computing the double summation of eq (A.2) is to perform first the finite sum at fixed $l = m + n$, and then the infinite sum over l .

$$\begin{aligned} \langle I_b | \sin^2(gH_b) | I_b \rangle &= \\ \frac{1}{2} \left[1 - N^2 \sum_l \frac{1}{2^l l!} |F_b|^2 \sum_{n=0}^l \binom{l}{n} \exp[2ig(l-n)] \exp[-2ign] \right] &= \\ \frac{1}{2} (1 - \exp[|F_b^2|(1 - \cos(2g))]) &. \end{aligned}$$

The last step requires the insertion of the actual value of the normalizing factor N , as it is given in eq (3.1'); then using eq (A.1) and the definition $|F_b^2| = T_b$ we get eq (4.3).

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